Fractional Programming

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Abstract

Fractional Programming, one of the various application of non linear programming, is applicable in various fields such as Finance and Economics, of which production planning, financial and corporate planning, health care, hospital planning are some of the application examples in Fractional Programming. In general, minimization or maximization of objective functions such as productivity, return on investment, return/risk, time /cost or output/input under limited constraint are some other examples of the applications of Fractional Programming. This paper focused on how to solve Fractional Programming Problem in particular on Linear Fractional Programming.

Fractional Programming

1 Introduction

In various application of non linear programming a ratio of two functions is to be maximized or minimized. In other applications the objective function involves more than one ratio of function. Ratio optimization problems are commonly called Fractional Programming. Some of the best examples of fractional programming are given in the form as,

$$\text{Efficiency} = \frac{\text{Technical Terms}}{\text{Economical Terms}}$$

$$\text{Growth Rate of an Economy} = \text{Max}_{x\in S}[\text{Min}_{1\leq i\leq m}\frac{\text{out put}_i}{\text{input}_i}]$$

where S denote a feasible production plane of the economy

Fractional programming deals with the optimization of one or several ratios of extended real-valued functions to constraints S given by,

$$f(x) \to \text{ Min}, x \in S \qquad (P)$$

Where $f(x)$ is a ratio of one or several ratios of extended real-valued functions like $f(x) = \frac{g(x)}{h(x)}$, $f(x) = \sum_{i=1}^{m} \frac{g_i(x)}{h_i(x)}$ or $f(x) = (\frac{g_1(x)}{h_1(x)}, \cdots, \frac{g_m(x)}{h_m(x)})$. Here both functions on the numerator and denominator are extended real-valued functions and finite valued on a feasible set S of (P).
Minimizations of cost-time ratio, maximization of out put-input ratio and maximization of profit-capital ratio or profit-revenue ratio are some other examples of fractional programming problem.

Now a day's, different researchers in their study shows different methods of solving fractional programming problems. In 1963, Gilmore and Gomory show how linear fractional programs can be solved with an adjacent vertex by following procedures just like linear programs with simplex method. Separately, in 1962 Charnes and Cooper show how a linear fractional program can be reduced to a linear program by using a non-linear variable transformation.
Fractional programs with one or more ratios have often been studied in the broader context of generalized convex programming. Ratios of convex and concave functions as well as composites of such ratios are not convex in general, even in the case of linear ratios, but often they are generalized as a convex. Fractional programming also overlaps with global optimization.
Now we are going to see different classification of fractional programming based on the nature of the function on the numerator, denominator and the constraint. The purpose of the following overview is to demonstrate the diversity of the problem.
Here after on this paper we use minimization of the problem, since maximization problem is the same as the minimization of the negative of the same problem.

2 Classification of Fractional Programming

1. Single-Ratio Linear Fractional Programming
   This kind of problem is given by

   $$\frac{g(x)}{h(x)} \rightarrow \text{Min},\ x \in S \qquad (P)$$

   Where $S = \{x : l_k(x) \leq 0, k = 1 : n\}$, $h(x) > 0$ for all $x \in S$ and $l_k : x \rightarrow R$, where $k = 1 : n$ and g, h and l_k are affine functions (linear plus a constant)

2. Single-Ratio Fractional Programming
   This kind of problem is given by

   $$\frac{g(x)}{h(x)} \rightarrow \text{Min},\ x \in S \qquad (P)$$

   Where g and h are extended real-valued functions which are finite valued on S, and $h(x) > 0$ for all $x \in S$ and S is non empty closed feasible region in X.

3. Single-Ratio Quadratic Fractional Programming
   This kind of problem is given by

   $$\frac{g(x)}{h(x)} \rightarrow \text{Min},\ x \in S \qquad (P)$$

   Where g and h are quadratic, $h(x) > 0$ for all $x \in S$ and $l_k : x \rightarrow R$ where $k = 1 : n$ is affine functions.

4. Generalized Fractional Programming
   This kind of problem is given by

   $$\text{Sup}_{1 \leq i \leq m} \frac{g_i(x)}{h_i(x)} \rightarrow \text{Min},\ x \in S \qquad (P)$$

   With extended real-valued functions $g_i, h_i : X \rightarrow [-\infty, \infty]$, which are finite valued on S with $h_i(x) > 0$ for every $i = 1 : m$, and $x \in S$.

5. Min-Max Fractional Programming
   This kind of problem is given by

   $$\text{Min}_{x \in S} \text{Max}_{y \in W} \frac{g(y, x)}{h(y, x)} \qquad (P)$$

Where $S \subset R^m$ and $W \subset R^n$ are non empty closed set and $g : R^{m+n} \to [-\infty, \infty]$ is a finite-valued function on $S \times W$. In case $h : R^{m+n} \to [-\infty, \infty]$ is a finite- valued positive function on $S \times W$.

6. Sum-of-Ratio Fractional Programming
   This kind of problem is given by

$$\sum_{i=1}^{m} \frac{g_i(x)}{h_i(x)} \to \text{Min} \ ,x \in S \qquad (P)$$

   With $h_i(x) > 0$ for every $i = 1 : m$ and $x \in S$.

7. Multi-Objective Fractional Programming
   This kind of problem is given by

$$(\frac{g_1(x)}{h_1(x)}, \cdots, \frac{g_m(x)}{h_m(x)}) \to \text{Min} \ ,x \in S \qquad (P)$$

   With $h_i(x) > 0$ for every $i = 1 : m$ and $x \in S$.

The remaining paper is focus on linear fractional programming and methods of solving linear fractional programming since it is one of a particular type of fractional programming.

3 Linear Fractional Programming

Problems in which the objective function is the ratio of two affine functions and the constraints are affine inequality are called linear fractional programming problems. Such programming problems have recently been a subject to wide interest in non linear programming. One of a particular example of linear fractional programming is stack cutting problem, Gilmore and Gomory discuss a stack cutting problems in the paper industry for which under the given circumstance it is more appropriate to minimize the ratio of wasted and used amount of raw material rather than just minimizing the amount of wasted materials. This stack cutting problem is formulating as a linear fractional program, since production is naturally experienced as a linear function.
Linear fractional programming can be stated precisely as

$$\begin{aligned} \text{Minimum} \quad & f(x) = \frac{\sum_{j=1}^{l} c_j x_j + \alpha}{\sum_{j=1}^{l} d_j x_j + \beta} \qquad (1) \\ \text{Subject to} \quad & \sum_{j=1}^{l} a_{ij} x_j \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} b_i \\ & x_j \geq 0 \\ & \text{where } i = 1 : m \text{ and } j = 1 : l \end{aligned}$$

It is assumed that the constraint set S given by (1) is regular, i.e the set of feasible solution is not empty and bounded, and the denominator of the objective function is strictly positive for all feasible solutions. In order to solve this linear fractional programming, we must first convert it in to which is known as standard form as

$$\begin{aligned} \text{Minimum} \quad & f(x) = \frac{c^t x + \alpha}{d^t x + \beta} \qquad (P) \\ \text{Subject to} \quad & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ & x_j \geq 0 \\ & \text{where } i = 1 : m \text{ and } j = 1 : l \end{aligned}$$

and the number of variables n may or may not be same as before.
The process of conversation may require several steps.

- Step 1: If the linear fractional program originally formulated as the maximization of $f(x)$, we can instead substitute the equivalent objective to minimization of $-f(x)$.
- Step 2: If any variable x_j is not restricted to non negative values it can be eliminated by transformation $x_j = x_j^* - x_j^{**}$,where $x_j^* - x_j^{**} \geq 0$. Every real valued of x_j can be expressed by non negativity values of x_j^* and x_j^{**}.
- Step 3: Finally, any inequality constraints in the original formulation can be converted to equations by the addition of non negativity slack or surplice variables. Thus, the constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

  Would become

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{l+1} \qquad\quad &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad\quad - x_{l+2} &= b_2 \\ \text{with } x_{l+1}, x_{l+2} \geq 0 \end{aligned}$$

  Here x_{l+1} and x_{l+2} are slack and surplus variables, respectively.

In matrix or vector notation, the standard form of the linear fractional programming problems is written as

$$\begin{array}{lll} \text{Minimum} & f(x) = \dfrac{c^t x + \alpha}{d^t x + \beta} & (P) \\ \text{Subject to} & Ax = b & \\ & x \geq 0 & \end{array}$$

Where c and d are n vectors, b and x are columns of vectors having m and n components, respectively. In addition to this, A is an $m \times n$ matrix, α and β are real scalars.

Example 3.1 *Activity Analysis to Maximize Rate of Return.*
There are l activities $x_1, x_2, \cdots, x_l$ a company may employ using the available supply of m resources $R_1, R_2, \cdots, R_m$. Let b_i be the available supply of R_i and let a_{ij} be the amount of R_i used in operating x_j at unit intensity. Let c_j be the net return to the company for operating x_j at unit intensity, and let d_j be the time consumed in operating x_j at unit intensity. Certain other activities not involving $R_1, R_2, \cdots, R_m$ are required of the company and yield net return α at time consumption of β. The problem is to maximize the rate of return $\frac{c^t x+\alpha}{d^t x+\beta}$ subject to the restrictions $Ax = b$ and $x \geq 0$. We note that the constraints set is non empty if $b \geq 0$, that it is generally bounded (for example if $a_{ij} > 0$ for all i and j) and that $d^t x + \beta$ is positive on the constraint set if $d \geq 0$ and $\beta > 0$.

Now let's see some important nature of the objective functions of linear fractional programming.

Definition 3.2 *Let S be a subset of a real linear space.*

1. *The set S is called convex if for all $x, y \in S$, then $\lambda x + (1 - \lambda)y \in S$ for all $\lambda \in (0, 1)$*
2. *Let the set S be a non-empty and convex. A functional $f : S \to R$ is called convex if for all $x, y \in S$, then*
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \text{ for all } \lambda \in (0, 1)$$

3. *Let the set S be a non-empty and convex, and a functional $f : S \to R$ is called concave. If the function $-f$ is convex.*

Definition 3.3 *Let $f : S \to R$, where S is a non-empty convex subset of a real linear space. The function f is said to be quasiconvex. If for each $x_1, x_2 \in S$, the following inequality is true.*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \textit{maximum}\{f(x_2), f(x_1)\} \textit{ for all } \lambda \in (0,1)$$

The function f is said to be quasiconcave if $-f$ is quasiconvex.

The optimal solution of a function having quasiconcave property occurs at the extreme points of the polyhedral set.

Definition 3.4 *Let $f : S \to R$, where S is a non-empty convex subset of a real linear space. The function f is said to be strict quasiconvex. If for each $x_1, x_2 \in S$ with $f(x_2) \neq f(x_1)$, we have*

$$f(\lambda x_1 + (1-\lambda)x_2) < \textit{maximum}\{f(x_2), f(x_1)\} \textit{ for all } \lambda \in (0,1)$$

The function f is said to be strictly quasiconcave if $-f$ is strictly quasiconvex.

Definition 3.5 *Let the set S be a non empty convex subset of a real linear space and let $f : S \to R$ be a given functional which has a directional derivative at some $x^* \in S$ in every direction $x - x^*$ with arbitrary $x \in S$. The functional f is called pseudoconvex at x^* if for all $x \in S$,*

$$(x - x^*)\nabla f(x^*) \geq 0 \Rightarrow f(x) - f(x^*) \geq 0$$

Or equivalently if

$$f(x) - f(x^*) \leq 0 \Rightarrow (x - x^*)\nabla f(x^*) \leq 0$$

The function f is said to be pseudoconcave if $-f$ is pseudoconvex.

Pseudoconvex, strictly quasiconvex, quasiconvex and convex functions are various types of function that share some desirable properties such every local minimal point also a global minimal point.

Theorem 3.6 *Let S be a non-empty open subset of real linear space and $f : S \to R$ be differentiable pseudoconvex function on S, then f is both strictly quasiconvex and quasiconvex.*

Proof: We first show that $f(x)$ is strictly quasiconvex.
We can prove this by contradiction. Suppose that there exist $x_1, x_2 \in S$ such that $f(x_1) \neq f(x_2)$ and

$$f(x_\lambda) \geq \max\{f(x_1), f(x_2)\} \text{ where } x_\lambda = \lambda x_1 + (1-\lambda)x_2 \text{ for all } \lambda \in (0,1)$$

with out loss of generality, assume that $f(x_1) < f(x_2)$, so that

$$f(x_\lambda) \geq f(x_2) > f(x_1) \qquad (1)$$

Because of pseudoconvexity of $f(x)$, $\nabla f(x_\lambda)'(x_1 - x_\lambda) < 0$.
Now, since $\nabla f(x_\lambda)'(x_1 - x_\lambda) < 0$ and $x_1 - x_\lambda = \frac{-(1-\lambda)(x_2-x_1)}{\lambda}$, then $\nabla f(x_\lambda)'(x_2 - x_\lambda) > 0$
By pseudocovexity of $f(x)$, we must have

$$f(x_2) \geq f(x_\lambda) \qquad (2)$$

By (1) and (2), we get $f(x_2) = f(x_\lambda)$.
Also, since $\nabla f(x_\lambda)'(x_2 - x_\lambda) > 0$, there exist a point $x_\eta = \eta x_1 + (1-\eta)x_2$ with $\eta \in (0,1)$, such that $f(x_\eta) > f(x_\lambda) = f(x_2)$. Again, by pseudoconvexity of $f(x)$, we have $\nabla f(x_\eta)'(x_2 - x_\eta) < 0$.
Similarly $\nabla f(x_\eta)'(x_\lambda - x_\eta) < 0$. Summarizing, we must have

$$\nabla f(x_\eta)'(x_2 - x_\eta) < 0 \qquad (3)$$
$$\nabla f(x_\eta)'(x_\lambda - x_\eta) < 0 \qquad (4)$$

But $x_2 - x_\eta = \frac{\eta(x_\eta - x\lambda)}{(1-\eta)}$ and, hence inequality (3) and (4) are not compatible. This contradicts that $f(x)$ is strictly quasiconvex.
Now we have to show $f(x)$ is also quasiconvex.
Let $x_1, x_2 \in S$ and $f(x)$ is a lower semi-continuous. If $f(x_1) \neq f(x_2)$, then, by the strict quasiconvexity of $f(x)$, we must have $f(x_\lambda) < \max\{f(x_1), f(x_2)\}$ where $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ for some $\lambda \in (0,1)$
Now suppose that $f(x_1) = f(x_2)$, to show that $f(x)$ is quasiconvex, we need to show that $f(x_\lambda) \leq f(x_1)$ for each $\lambda \in (0,1)$. By contradiction, suppose that $f(x_\eta) > f(x_1)$ where $x_\eta = \eta x_1 + (1-\eta)x_2$ for some $\eta \in (0,1)$, since $f(x)$ is lower semi-continuous, there exist $\lambda \in (0,1)$ such that

$$f(x_\eta) > f(x_\lambda) > f(x_1) = f(x_2) \qquad (5)$$

Note that x_η can be represented as a convex combination of x_λ and x_2. Hence, by the strict quasiconvexity of $f(x)$ and since $f(x_\lambda) > f(x_2), f(x_\eta) < f(x_\lambda)$, This contradicting to (5). This completes the proof.
Strictly quasiconvex is important in non-linear programming, which assures that a local minimum over a convex set is also global minimum and quasiconvex is also important in non-linear programming, which assures the existence of optimal solution at the extreme point if the solution is exist.

Lemma 3.7 *Let $f(x) = \frac{c^t x + \alpha}{d^t x + \beta}$, and let S be a convex set such that $d^t x + \beta \neq 0$ over S, then $f(x)$ is both pseudoconvex and pseudoconcave.*

Proof: Since $d^t x + \beta \neq 0$, either $d^t x + \beta > 0$ for all $x \in S$ or $d^t x + \beta < 0$ for all $x \in S$. Otherwise, there exist $x_1, x_2 \in S$ such that $d^t x_1 + \beta > 0$ and $d^t x_2 + \beta < 0$.
Hence, for some convex combination x of x_1 and x_2, $d^t x + \beta = 0$, this contradiction our assumption.
First we have to show f is pseudoconvex
Suppose that $x_1, x_2 \in S$ with $(x_2 - x_1)^t \nabla f(x_1) \geq 0$, we need to show that $f(x_1) \geq f(x_2)$. We have

$$\nabla f(x_1) = \frac{(d^t x_1 + \beta)c - (c^t x_1 + \alpha)d}{(d^t x_1 + \beta)^2}$$

Since $(x_2 - x_1)^t \nabla f(x_1) \geq 0$ and $(d^t x + \beta)^2 > 0$, it follows

$$\begin{aligned} 0 &\leq (x_2 - x_1)^t((d^t x_1 + \beta)c - (c^t x_1 + \alpha)d) \\ 0 &\leq (c^t x_2 + \alpha)(d^t x_1 + \beta) - (d^t x_2 + \beta)(c^t x_1 + \alpha) \end{aligned}$$

Therefore, $(c^t x_2 + \alpha)(d^t x_1 + \beta) \geq (d^t x_2 + \beta)(c^t x_1 + \alpha)$

But, since $d^t x_1 + \beta$ and $d^t x_2 + \beta$ are both either positive or negative, dividing by $(d^t x_1 + \beta)(d^t x_2 + \beta) > 0$, We get

$$\frac{c^t x_2 + \alpha}{d^t x_2 + \beta} \geq \frac{c^t x_1 + \alpha}{d^t x_1 + \beta}, \text{i.e } f(x_2) \geq f(x_1)$$

Therefore f is pseudoconvex.
Now we have to show f is pseudoconcave.
Suppose that $x_1, x_2 \in S$ with $(x_2 - x_1)^t \nabla f(x_1) \leq 0$, we need to show that $f(x_2) \leq f(x_1)$
We have

$$\nabla f(x_1) = \frac{(d^t x_1 + \beta)c - (c^t x_1 + \alpha)d}{(d^t x_1 + \beta)^2}$$

Since $(x_2 - x_1)^t \nabla f(x_1) \leq 0$ and $(d^t x + \beta)^2 > 0$, it follows

$$\begin{aligned} (x_2 - x_1)^t((d^t x_1 + \beta)c - (c^t x_1 + \alpha)d) &\leq 0 \\ (c^t x_2 + \alpha)(d^t x_1 + \beta) - (d^t x_2 + \beta)(c^t x_1 + \alpha) &\leq 0 \\ \text{Therefore, } (c^t x_2 + \alpha)(d^t x_1 + \beta) \leq (d^t x_2 + \beta)(c^t x_1 + \alpha) & \end{aligned}$$

But, since $d^tx_1+\beta$ and $d^tx_2+\beta$ are both either positive or negative, dividing by $(d^tx_1+\beta)(d^tx_2+\beta) > 0$, We get

$$\frac{c^tx_2+\alpha}{d^tx_2+\beta} \leq \frac{c^tx_1+\alpha}{d^tx_1+\beta} \text{i.e } f(x_2) \leq f(x_1)$$

Therefore f is pseudoconcave.
Therefore linear fractional programming is both pseudoconvex and pseudoconcave property. This completes the proof.
By definition of pseudoconvexity and pseudoconcavity, every local optimal solution is also a global optimal solution. In other word by Theorem 5, because of strictly quasiconvex and strictly quasiconcave we can ensure that a local optimal solution is also a global optimal solution.

Theorem 3.8 *Consider the problem:*

$$\begin{aligned} &\textit{Minimum} && f(x) = \frac{c^tx+\alpha}{d^tx+\beta} \\ &\textit{Subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Where c and d are n vectors, b and x are columns of vectors having m and n components, respectively, A is an $m \times n$ matrix, α and β are real scalars, then the optimal solution occur at extreme point.

Proof: Suppose the minimum of $f(x)$ is attain at $x_0 \in S = \{x : Ax = b, x \geq 0\}$. If there is an extreme point whose objective is equal to $f(x_0)$, then the result is at hand. Otherwise, let $x_1, x_2, \cdots, x_k$ be the extreme points of S, and assume that $f(x_0) < f(x_j)$ for $j = 1 : k$. So, x_0 can be represented as

$$x_0 = \sum_{j=1}^{k} \lambda_j x_j,\ \sum_{j=1}^{k} \lambda_j = 1,\ j = 1 : k$$

Since $f(x_0) < f(x_j)$ for each j, then

$$f(x_0) < \min f(x_j) = \alpha \qquad (1)$$

Now consider the upper level set $S_\alpha = \{x : f(x) \geq \alpha\}$. From (1) $x_j \in S_\alpha$ for $j = 1 : k$ and by quasicocavity of $f(x)$, S_α is convex. Hence $x_0 = \sum_{j=1}^{k} \lambda_j x_j$ belongs to S_α. This implies that, $f(x_0) \geq \alpha$. Which contradicts (1).This contradiction shows that $f(x_0) = f(x_j)$ for some extreme point x_j, and the proof is complete.
Now we can summarize fractional programming as follows, the optimal solution for a linear fractional program occurs at extreme points of the feasible region. Furthermore, every local minimum is also a global minimum because of convexity property.

4 Methods of Solving Linear Fractional Programming

We have different kinds of methods for solving linear fractional programming. The first method is convex-simplex methods; this method is a minor modification of simplex method of linear programming. The other method is a procedure credited by Gilmore and Gomory for solving a linear fractional program. In addition to this, Charnes and Cooper describe another procedure of solving linear fractional programming by reducing the problem in to a linear programming by using non-linear variable transformation. Now we have to see each of the methods one by one.

4.1 Convex-Simplex Method

Convex simplex method (Zangwill 1967) is used to solve the optimal value of convex objective function subject to linear inequality constraints. However, the method is applicable to problems where the objective function is a general function having continuous partial derivatives. Because of the special structure

of the objective function of linear fractional programming, convex simplex method is a true generalization of linear simplex method both in sprite and in the fact that the same tableau and variable selection techniques are used. If the objective function is linear, then the convex simplex method reduced to the linear simplex method. Moreover, the convex simplex method is actually behaves like the linear simplex method whenever it encounters a linear portion of a convex objective function.
Convex simplex method coincides with the linear simplex method when applied to linear fractional programming problems in the case when we initially start with a basic feasible solution to the problem. But the Convex simplex method will differ from linear simplex method in linear fractional programming if we start with any not basic feasible solution to the problem at least for the steps. Once we reach the basic feasible solution, to start with the steps are just the same as linear simplex method for the next iteration.
Now our first task is to pose conditions under which a given point x* is optimal by restating kuhn-Tuckre condition

Karush-Kuhn-Tucker (KKT) conditions

Consider problem

$$\begin{array}{lll} \text{Minimum} & f(x) & (P) \\ \text{Subject to} & Ax = b & \\ & x \geq 0 & \end{array}$$

Where $f(x)$ is a convex function with continuous first partial derivatives, A is an $m \times n$ matrix of rank m, b and x are column vectors of m and n components, respectively.
First we change the problem in to unconstraint problem by using lagrange multiplier and the problem become

$$L(x, \lambda, \mu) = f(x) - \lambda^t(Ax - b) - \mu^t x, \text{where } \lambda \in R^m,\ \mu \in R^n$$

$$\begin{aligned} L_x(x, \lambda, \mu) &= \nabla f(x) - \lambda^t A - \mu^t = 0 \Rightarrow \nabla f(x) - \lambda^t A = \mu^t && (1) \\ L_\lambda(x, \lambda, \mu) &= Ax - b = 0 \Rightarrow Ax = b && (2) \\ L_\mu(x, \lambda, \mu) &= -x \leq 0 \Rightarrow -x \leq 0 && (3) \\ \mu^t x &= 0 \text{ and} && (4) \\ \mu &\geq 0 && (5) \\ & && (6) \end{aligned}$$

From (1) and (5) we have

$$\nabla f(x)^t - \lambda^t A = \mu^t \geq 0 \Rightarrow \nabla f(x)^t - \lambda^t A \geq 0 \qquad (*)$$

From (1) and (4) we have

$$(\nabla f(x)^t - \lambda^t A)x = 0 \qquad (**)$$

From (7)

$$\begin{aligned} (\nabla_B f(x)^t - \lambda^t B, \nabla_N f(x)^t - \lambda^t N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} &= 0 \\ \Rightarrow (\nabla_B f(x)^t - \lambda^t B)x_B + (\nabla_N f(x)^t - \lambda^t N)x_N &= 0 \\ \Rightarrow (\nabla_N f(x)^t - \lambda^t N)x_N &= 0 \text{ since } x_N = 0 \end{aligned}$$

From the above

$$\begin{aligned} (\nabla_B f(x)^t - \lambda^t B)x_B &= 0 \text{ and } x_B > 0 \\ \Leftrightarrow \nabla_B f(x)^t - \lambda^t B &= 0 \\ \lambda^t = \nabla_B f(x)^t B^{-1} & \qquad (***) \end{aligned}$$

Substituting (***) in both equation (*) and (**), then we have

$$(\nabla f(x)^t - \nabla_B f(x)^t B^{-1} A) \geq 0 \text{ dual feasibility}$$
$$(\nabla f(x)^t - \nabla_B f(x)^t B^{-1} N)x = 0 \text{ complementary slackness condition}$$

We can write the above condition component wise as follows

$$(\frac{\partial f(x)}{\partial x_j} - \nabla_B f(x)^t B^{-1} a_j) \geq 0 \, j = 1 : n \text{ dual feasibility}$$
$$(\frac{\partial f(x)}{\partial x_j} - \nabla_B f(x)^t B^{-1} a_j) x_j = 0 \, j = 1 : n \text{ complementary slackness condition}$$

A point satisfying dual feasibility and complimentary slackness condition is a KKT point. Because of the convexity of the objective function $f(x)$, a point satisfying the KKT condition for a minimization problem is also a global minimum over the feasible region.
Now our first task is to pose conditions under which a given point x^* is optimal. Lemma 3. formulate the optimality conditions by restating kuhn-Tuckre condition.

Lemma 4.1 *Let A be any linear programming tableau with b the corresponding right hand side so that $Ax = b$, $x \geq 0$ if and only if x is feasible. Let x^* be a particular feasible point and the corresponding relative cost vector is $C(x^*)^t = (\nabla f(x^*)^t - \nabla_B f(x^*) B^{-1} A)$.*
Let

$$\begin{aligned} \alpha &= min\{\frac{\partial f}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j, j \in \overline{A}\} \text{ and} \\ \beta &= min\{\frac{\partial f}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j) x_j^*, j \in \overline{A}\} \end{aligned}$$

If $\alpha = \beta = 0$, then x^* is optimal.

Proof: Observe that the problem

$$\begin{array}{ll} \text{Minimum} & f(x) \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Since x^* is feasible, x^* must satisfy

$$\begin{aligned} Ax^* &= b \\ x^* &\geq 0 \end{aligned}$$

If $\alpha = \beta = 0$, x^* satisfies

$$\begin{aligned}
\text{If } \alpha &= 0 \\
&\Rightarrow \max\{\frac{\partial f}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j\} = 0 \\
&\Rightarrow \frac{\partial}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j \geq 0 \qquad (1) \\
\text{If } \beta &= 0 \\
&\Rightarrow \max\{(\frac{\partial f}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j) x_j^*\} = 0 \\
&\Rightarrow (\frac{\partial f}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j) x_j^* \leq 0 \\
&\quad \text{but by (1) } \frac{\partial f}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j \geq 0 \text{ and } x_j \geq 0 \\
&\Rightarrow (\frac{\partial f}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j) x_j^* = 0
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\frac{\partial f(x^*)}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j) \geq 0 \, j = 1 : n \text{ dual feasibility} \\
&(\frac{\partial f(x^*)}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j) x_j^* = 0 \, j = 1 : n \text{ complementary slackness condition}
\end{aligned}$$

A point satisfying dual feasibility and complimentary slackness condition are simply the karusa-kuhn-Tuckre point and because of convexity property of the objective function, a point indeed an optimal point. This completes the proof.
Now consider the problem

$$\begin{aligned}
\text{Minimum} \quad & f(x) \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}$$

Where $f(x)$ is a convex function with continuous first partial derivative, A is $m \times n$ matrix of rank m, b and x are columns of vectors having m and n components, respectively. We can write the constraint set as $Ax = x_1 a_1 + x_2 a_2 + \cdots + x_j a_j + \cdots + x_n a_n$, where $x_1, x_2, \cdots, x_n \geq 0$ and a_j is the j^{th} column of A, and assume that the rows of A are linearly independent. Suppose that m linearly independent columns $a_1^*, a_2^*, \cdots, a_m^*$ have selected from A. If we set the $n-m$ variables not associated with these columns equal to zero ,then the unique solution to the resulting system of m equations in the m unknowns $x_1^*, x_2^*, \cdots, x_m^*$ is called a basic solution. The columns $a_1^*, a_2^*, \cdots, a_m^*$ are the basic columns, where $x_1^*, x_2^*, \cdots, x_m^*$ are the basic variables.
If any one or more of basic variables has the values zero, then the basic solution is said to be degenerate. So we can be decompose A in to $[B, N]$, B is $m \times m$ matrix of full rank and N is an $m \times (n-m)$ matrix. i.e $B = [a_1^*, a_2^*, \cdots, a_m^*]$ and N is the remaining $m \times (n-m)$ matrix.A is an mxn matrix, b and x are columns of vectors having m and n components respectively. So, A decompose in to B and N and x

decompose in to x_B and x_N, then we have

$$\begin{aligned}
Bx_B + Nx_N &= b \\
Bx_B &= b - Nx_N = b - \sum_{j \in N} a_j x_j \\
x_B &= B^{-1}b - \sum_{j \in N} B^{-1} a_j x_j \\
x_B &= \bar{b} - \sum_{j \in N} y_j x_j, \text{ where } \bar{b} = B^{-1}b \text{ and } y_j = B^{-1}a_j
\end{aligned}$$

We can write this component wise as follows

$$x_{B_i} = \overline{b_i} - \sum_{j \in N} y_{ij} x_j, \text{ for all } i = 1 : m \qquad (1)$$

Let $z = f(x_B, x_N)$. The partial derivative of z with respect to the nonbasic variables x_j, $l \in \overline{N}$, can be calculated by the chain rule as follows

$$\begin{aligned}
\frac{\partial z}{\partial x_j} &= \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \frac{\partial f}{\partial x_{B_i}} \frac{\partial x_{B_i}}{\partial x_j} \\
\frac{\partial z}{\partial x_j} &= \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \frac{\partial f}{\partial x_{B_i}} y_{ij} \text{ since } \frac{\partial x_{B_i}}{\partial x_j} = -y_{ij} \\
\frac{\partial z}{\partial x_j} &= \frac{\partial f}{\partial x_j} - \nabla_B f(x) y_j \qquad (2)
\end{aligned}$$

Where

- x_B vectors corresponding to B. i.e $x_B = (x_1^*, x_2^*, \cdots, x_m^*)^t$
- x_N vectors corresponding to N.
- $\overline{A}$ is the set of subscripts of the columns of matrix A, i.e $j \in \overline{A}$ if and only if x_j is the j^{th} component of x.
- $\overline{B}$ is the set of subscripts of the basic variables,i.e $j \in \overline{B}$ if and only if x_j is in x_B.
- $\overline{N}$ is the set of subscripts of the non basic variables,i.e $j \in \overline{N}$ if and only if x_j is in x_N.
- B_i the subscript i indicates the i^{th} component of the basic variables.
- $\nabla_B f(x)$ is the gradient of $f(x)$ with respect to the basic vector x_B.
- $\nabla_N f(x)$ is the gradient of $f(x)$ with respect to the non basic vector x_N.
- $(\nabla f(x))^t$ the superscript t indicates the transpose of $\nabla f(x)$.

Equation (2) is called relative cost vector $C(x)$ or reduced gradient vector. If the objective function is linear i.e $f(x) = c^t x$ equation (2) become $\triangle z_j = c_j - c_B y_j$ is called relative cost or reduced cost.
In the convex simplex method ,the partial derivative changes as x_j increases and may at some point fall to zero. When this happens it is no longer desirable to increase x_j even though it may still be feasible to do so. Accordingly, displacement ends either when

1. The value of some basic variable x_{B_r} falls to zero or
2. The partial derivative $\frac{\partial z}{\partial x_j}$ vanishes.

The point at which one of these two events first occurs is taken as the starting point of the next iteration. Thus the convex simplex method is a member of the feasible directions family. Here, it will be better to start with a basic feasible solution instead of any feasible solution to the problem because it is known that optimum of such problem occurs at the basic feasible solution to the problem. It would lead to lesser computational work. Also for such problems local optimum is global.

The Algorithmic Procedure

- Initialization step
  Find a starting basic feasible solution x^1 to the system of $Ax = b, x \geq 0$ from simplex and, go to step 1 of iteration k with $k = 1$.

- Iteration k
  The feasible point x^k and tableau A^k are given, where A^k is the value of a matrix A at k iteration which is also equivalent to matrix A.

  – Step 1: Calculates the relative cost vector(C).

  $$C(x^*)^t = (\bigtriangledown f(x^*)^t - \bigtriangledown f(x^*)_B Y)$$

  Where

  $$\begin{aligned} Y &= B^{-1}A^k \\ \bigtriangledown f(x^k) &= (\frac{\partial f(x^k)}{\partial x_1^k}, \frac{\partial f(x^k)}{\partial x_2^k}, \cdots, \frac{\partial f(x^k)}{\partial x_n^k})^t \\ \bigtriangledown f(x^k)_B &= (\frac{\partial f(x)}{\partial x_{B_1}^k}, \frac{\partial f(x)}{\partial x_{B_2}^k}, \cdots, \frac{\partial f(x)}{\partial x_{B_n}^k})^t \end{aligned}$$

  Let

  $$\begin{aligned} \alpha^k &= \min\{\frac{\partial f}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j, j \in \overline{A}\} \text{ and} \\ \beta^k &= \min\{(\frac{\partial}{\partial x_j^*} - \nabla_B f(x^*)^t B^{-1} a_j) x_j^*, j \in \overline{A}\} \end{aligned}$$

  If $\alpha^k = \beta^k = 0$, terminate x^k is optimal. Otherwise, go to step 2.

  – Step 2: Determine the non-basic variable to change. Let

  $$\begin{aligned} p \text{ is an index such that } \alpha^k &= \frac{\partial f}{\partial x_p^k} - \nabla_B f(x^k) y_p \text{ and} \qquad (1) \\ q \text{ is an index such that } \beta^k &= (\frac{\partial}{\partial x_q^k} - \nabla_B f(x^k) y_q) x_q^k \qquad (2) \end{aligned}$$

  If $|\alpha^k| \geq \beta^k$, increase x_p from (1) adjusting only basic variables.
  If $|\alpha^k| \leq \beta^k$, decrease x_q from (2) adjusting only basic variables.

  – Step 3: Calculate x^{x+1} and A^{k+1}

    * Case 1: x_p^k is to be increased and for some i, $y_{ip}^k > 0$.
      Increase x_p^k will drive a basic variable to zero. Let $z^k = (z_i^k)^t$ be the x valued when that occurs. Specifically

      $$\begin{aligned} z_i^k &= x_i^k, i \in \overline{N} - p \\ z_p^k &= x_p^k + \triangle^p \qquad (3) \\ z_{B_i}^k &= x_{B_i}^k - y_{ip}^k \triangle^k, i \in \overline{B} \end{aligned}$$

Where $\overline{N}-p$ is the set of subscripts of the non basic variables except p and $\overline{B}$ is the set of subscripts of the basic variables, and

$$\triangle^k = \frac{x^k_{B_r}}{y^k_{rp}} = \min\ \{\frac{x^k_{B_i}}{y^k_{ip}}; y^k_{ip} > 0\} \qquad (4)$$

Find x^{k+1}, where $f(x^{k+1}) = \min\{f(x)/x = \lambda x^k + (1-\lambda)z^k,\ 0 \leq \lambda \leq 1\}$
If $x^{k+1} \neq z^k$, set $A^{k+1} = A^k$, go to iteration k with $k+1$ replacing k. Do not change basis.
If $x^{k+1} = z^k$, pivot on y^k_{rp} and forming A^{k+1}, go to iteration k with $k+1$ replacing k, and replace $x^k_{B_r}$ with x^k_p in the basis.

* Case 2: x^k_p is to be increased and $y^k_{ip} < 0$ for all i.
Define z^k the same as in equation (3) except let $\triangle^k = 1$. Then attempt to determine x^{k+1} such that

$$f(x^{k+1}) = \min\{f(x)/x = x^k + \lambda(z^k - x^k),\ \lambda \geq 0\}$$

If no x^{k+1} exists, terminate. The optimal solution is unbounded.
If x^{k+1} does exist, set $A^{k+1} = A^k$, go to iteration k with $k+1$ replacing k and the same basis.

* Case 3: x^k_q is decrease determining z^k using equation (3) except defining $\triangle^k$ as

$$\triangle^k = \max\ [\triangle^k_1,\ \triangle^k_2]$$

Where $\triangle^k_1 = \frac{x^k_{b_r}}{y^k_{rq}} = \max\ \{\frac{x^k_{b_i}}{y^k_{iq}} / \, y^k_{iq} < 0\}$ and $\triangle^k_2 = -x^k_q$
If $y^k_{iq} \geq 0, i = 1:m,$ let $\triangle^k_1 = -\infty$
Find x^{k+1}, where $f(x^{k+1}) = \min\{f(x)/x = \lambda x^k + (1-\lambda)z^k,\ 0 \leq \lambda \leq 1\}$
If $x^{k+1} \neq z^k$, set $A^{k+1} = A^k$, go to iteration k with $k+1$ replacing k. Do not change basis.
If $x^{k+1} = z^k$, pivot on y^k_{rq} and forming A^{k+1}, go to iteration k with $k+1$ replacing k, and replace $x^k_{b_r}$ with x^k_q in the basis.

We continue this process with the new feasible solution x^{k+1}_B till we obtain that $\alpha^n = \beta^n = 0$ is satisfied at the n^{th} iteration which is the condition of optimality and hence,x^n will be the optimal solution and the corresponding optimal value of $f(x)$ will be $f(x^n)$.

Remark 4.2 *In order to calculate x^{k+1}, the next feasible solution to start with, we solve*

$$f(x^{k+1}) = min\ \{f(x)/x = \lambda x^k + (1-\lambda)y^k, 0 \leq \lambda \leq 1\}$$

But in the case of linear fractional programming problem, by theorem 2 the optimal solution is obtained at a basic feasible solution S and one of the basic feasible solutions to the problem is optimal. Therefore, it will always be true that $x^{k+1} = z^k$.

Theorem 4.3 *If the objective functions is linear fractional programming, then $\beta^k = 0$ for any iteration, where*

$$\beta^k = max\ \{(\frac{\partial}{\partial x^k_j} - \nabla_B f(x)y_j)x^k_j, j \in \overline{A}\}.$$

Proof: First we have to show a complimentary slackness condition for non basic variables.
Since $x^k_j = 0$, for all $j \in \overline{N}$ we have $(\frac{\partial f}{\partial x^k_j} - \nabla_B f(x)y_j)x^k_j$, for all $j \in \overline{N}$. Therefore, a complimentary slackness condition is hold for non basic variables x_N.
Now we have to proof a complimentary slackness condition for a basic variables x_B i.e $j \in \overline{B}$. If $x_j, j \in \overline{B}$,

then y_j must be the unit vector e_r.

$$\begin{aligned}
\frac{\partial z}{\partial x_j^k} &= \frac{\partial f}{\partial x_j^k} - \bigtriangledown_B f(x) y_j, \text{for all } j \in \overline{B} \text{ where } y_j = B^{-1}a_j \\
\frac{\partial z}{\partial x_j^k} &= \frac{\partial f}{\partial x_j^k} - \sum \frac{\partial f}{\partial x_{B_i}} y_{ij} \\
\frac{\partial z}{\partial x_j^k} &= \frac{\partial f}{\partial x_j^k} - (\frac{\partial f}{\partial x_{B_1}} y_{1j} + \cdots + \frac{\partial f}{\partial x_{B_i}} y_{ij} + \cdots + \frac{\partial f}{\partial x_{B_m}} y_{mj}) \\
\frac{\partial z}{\partial x_j^k} &= \frac{\partial f}{\partial x_j^k} - \frac{\partial f}{\partial x_{B_j}} y_{jj}, \text{since } y_j \text{is the unit vector} \\
\frac{\partial z}{\partial x_j^k} &= \frac{\partial f}{\partial x_j^k} - \frac{\partial f}{\partial x_j} = 0, \text{for all } j \in \overline{B} \\
&\Rightarrow \frac{\partial z}{\partial x_j} x_j = 0 \text{ for all } j \in \overline{B}
\end{aligned}$$

Therefore a complimentary slackness condition is hold for basic variables. Therefore complimentary slackness condition is hold i.e

$$(\frac{\partial f}{\partial x_j^k} - \nabla_B f(x^k) y_j) x_j^k = 0 \text{ for all } j \in \overline{A} \text{ and}$$

$$\beta^k = \max\{(\frac{\partial f}{\partial x_j^k} - \nabla_B f(x^k) y_j) x_j^k \text{ for all } j \in \overline{A}\} = 0.$$

This is the proof.

Corollary 4.4 *Consider the problem*

$$\begin{aligned}
&\textit{Minimum} \quad && f(x) = \frac{c^t x + \alpha}{d^t x + \beta} \\
&\textit{Subject to} && Ax = b \\
& && x \geq 0
\end{aligned}$$

Let A^k be any linear programming tableau with b^k the corresponding right hand side so that $A^k x^k = b^k, x^* \geq 0$, if and only if x^* is feasible. Let x^k be a particular feasible point and the corresponding relative cost vector is

$$C(x^k)^t = (\bigtriangledown f(x^k)^t - \bigtriangledown f(x^k)_B Y) \, Where \; Y = B^{-1} A^k$$

Let $\alpha^k = min \, \{\frac{\partial f}{\partial x_j^k} - \nabla_B f(x^k) y_j, j \in \overline{A}\}$
If $\alpha^k = 0$, then x^k is optimal.

Proof: If $\alpha^k = \beta^k = 0$, then x^k is optimal by lemma 3 and from above theorem we have $\beta^k = 0$ for any iteration k, therefore $\alpha^k = 0$ is the optimal condition for linear fractional programming problem.

Remark 4.5 *Suppose that we are given an extreme point of the feasible region with basis B such that $x_B = B^{-1}b > 0$ and $x_N = 0$. To get a lower objective function value needs to increase or decrease one of the non basic variables accordingly. Since the current point is an extreme point with $x_N = 0$ decreasing a non basic variable is not permitted as it would violate the nonnegative restriction.*

Gilmore and Gomory Procedure

Now we have to see below a procedure credited to Gilmore and Gomory (1963) for solving a linear fractional programming.

- Initialization step: Find a starting basic feasible solution x^1 to the system of $Ax = b$, $x \geq 0$ from simplex and, go to step 1 of iteration k with $k = 1$.
- Iteration k The feasible point x^k and tableau A^k are given.
  - Step 1: Compute the vector α^k.
    If $\alpha^k = 0$, terminate x^k is an optimal. Otherwise, go to step 2.
  - Step 2: Let p is an index such that $\alpha^k = \frac{\partial f}{\partial x_p^k} - \nabla_B f(x^k) y_p^k$ for which $\frac{\partial f}{\partial x_p^k} - \nabla_B f(x^k) y_p^k < 0$. This gives x_p^k is increasing from zero to some positive number. Now we can determine a basic variable which falls to zero.

$$x_{B_i}^k = \overline{b^k} - x_p^k y_{ip}^k > 0 \qquad (1)$$

    * Case 1: x_p^k is increasing and for some i, $y_{ip}^k > 0$.

$$\begin{aligned} \triangle^k &= \min \{ \frac{x_{B_i}^k}{y_{ip}^k}, \text{for some } y_{ip}^k > 0 \} \\ \triangle &= \frac{x_{B_r}^k}{y_{rp}^k} \qquad (2) \end{aligned}$$

      $\Rightarrow x_{B_r}^k$ is the new non basic variables which falls to zero. Up date the table corresponding pivoting at y_{rp}^k and by substituting (2) in to (1) we have

$$\begin{aligned} z_1^k &= x_i^k, i \in \overline{N} - p \\ z_p^k &= x_p^k + \triangle^k \qquad (3) \\ z_{B_i}^k &= x_{B_i}^k - y_{ip}^k \triangle^k, i \in \overline{B} \end{aligned}$$

      We have got a new basic feasible solution z^k and x^{k+1}, go to iteration k with $k+1$ replacing k, go to step 1.

    * Case 2: x_p^k is increasing and $y_{ip}^k < 0$ for all i. Here we can increase x_p^k as we need with out changing a basic variable to zero. The optimal solution is unbounded.

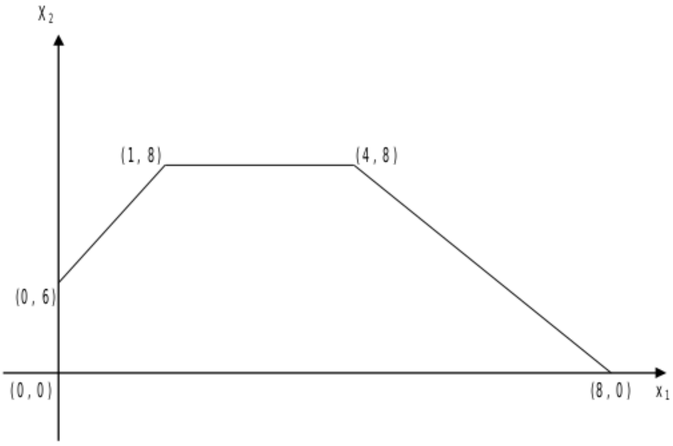
Remark 4.6 *We continue this process with the new feasible solution x^{k+1} till we obtain that α^k is satisfied at the n^{th} iteration which is the condition of optimality and hence, x^n will be the optimal solution and the corresponding optimal value of $f(x)$ will be $f(x^n)$.*

We now assume that $x_B > 0$ for each extreme point. This algorithm moves from one extreme point to another. By nondegeneracy assumption, objective function strictly decreases at each iteration, so that the extreme point generated is distinct. There are only a finite number of these points, and, hence, the algorithm stops in a finite number of steps. At the end, the relative cost vector is non negative i.e $\alpha^k = 0$ resulting in a Karush-Kuhn-Tucker (KKT) point and because of pseudoconvexity and pseudoconcavitiy. This point indeed an optimal point.
Now we can solve numerical example of linear fractional programming problem by using convex simplex algorithm.

Example 4.7 *Consider the following problem*

$$\begin{aligned} &\textit{Minimum} && \frac{-4x_1 + 3x_2 + 5}{2x_1 + 4x_2 + 6} \\ &\textit{Subject to} && -2x_1 + x_2 \le 6 \\ &&& x_2 \le 8 \\ &&& 2x_1 + x_2 \le 16 \\ &&& x_1, x_2 \ge 0 \end{aligned}$$

Solution: Before solving by using convex-simplex method, let we see what looks like the figure of the constraints Figure shows the feasible region with the extreme points $(0,0)$,$(0,6)$,$(1,8)$,$(4,8)$ and $(8,0)$.The



value objective function at those points is 0.833, 0.766, 0.625, 0.2826 and -1.227 respectively. Hence the optimal solution is $(8,0)$ and the corresponding optimal value is -1.227.
Now we have to solve this problem by using convex-simplex method, by introducing additional slack variables x_3, x_4 and x_5 the problem (P) is changed to standard linear fractional programming. The problem is given by

$$\begin{aligned} &\text{Minimum} && \frac{-4x_1 + 3x_2 + 5}{2x_1 + 4x_2 + 6} \\ &\text{Subject to} && -2x_1 + x_2 + x_3 \qquad\qquad = 6 \\ &&& \qquad x_2 + \qquad x_4 \qquad = 8 \\ &&& 2x_1 + x_2 + \qquad\qquad x_5 = 16 \\ &&& x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

- Iteration 1: Let $x^1 = (0,0,8,6,16)$ since $x_B = B^{-1}b > 0$. Here $x^1_B = (x_3, x_4, x_5) = (6,8,16)$.
- Step 1: $\triangledown f(x^1) = (\frac{-34}{25}, \frac{-2}{25}, 0, 0, 0)$ and then $C(x^1) = (\frac{-34}{25}, \frac{-2}{25}, 0, 0, 0)$. The corresponding tableau A^1 is given by

	$\triangledown f(x^1)$	$\frac{-34}{25}$	$\frac{-2}{25}$	0	0	0		
$\triangledown f(x^1_B)$	$x^1_B \nwarrow x^1$	x^1_1	x^1_2	x^1_3	x^1_4	x^1_5	b	$\triangle^1$
0	x^1_3	-2	1	1	0	0	6	-
0	x^1_4	0	1	0	1	0	8	-
0	x^1_5	2	1	0	1	1	16	8
	$C(x^1)$	$\frac{-34}{25}$	$\frac{-2}{25}$	0	0	0		

$\alpha^1 = \min\{\frac{-34}{25}, \frac{-2}{25}, 0, 0, 0\} = \frac{-34}{25}$ Since $\alpha^1 \neq 0$, we go to the next step

- Step 2: Let $p = 1$ since $\alpha^1 = c_1$ where c_1 is the 1^{st} component of C. We can increase x_1^1 since $\alpha^1 = \frac{-34}{25}$
- Step 3: Calculate x^2 and A^2
  Increase x_1^1 from 0 to $\triangle^1$ and one of a basic variable in x^1 has become zero.
  Let $z^1 = (z_i^k)^1$ be the x valued when that occurs specifically

$$\begin{aligned}
z_1^1 &= x_1^1 + \triangle^1, \text{where } \triangle^1 = \min\{\frac{16}{2}, 2 > 0\} \\
z_1^1 &= 0 + 8 = 8 \\
z_2^1 &= 0 \\
z_3^1 &= 6 - (-2)8 = 22 \\
z_4^1 &= 8 - (0)8 = 8 \\
z_5^1 &= 16 - (-2)8 = 0
\end{aligned}$$

So, $z^1 = (8, 0, 22, 8, 0)$
Now we have find x^2, where $f(x^2) = \min\ \{f(x)/\lambda x^1 + (1-\lambda)y^1,\ 0 \leq \lambda \leq 1\}$. Since the solution of linear fractional programming is occurs at the extreme point, the new iteration point is $x^2 = y^1 = (8, 0, 22, 8, 0)$.
We can pivot on y_{31}^1 and the corresponding tableau A^2 is given by

	$\nabla f(x^2)$	$\frac{-34}{25}$	$\frac{-2}{25}$	0	0	0		
$\nabla f(x_B^2)$	$x_B^2 \nwarrow x^2$	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2	b^2	$\triangle^2$
0	x_3^2	0	2	1	0	0	22	
0	x_4^2	0	1	0	1	0	8	
$\frac{-34}{25}$	x_1^2	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	
	$C(x^2)$	0	$\frac{-2}{25}$	0	0	$\frac{17}{484}$		

$$\alpha^2 = \min\{0, \frac{191}{484}, 0, 0, \frac{17}{484}\} = 0$$

Since $\alpha^2 = 0$. This is the condition of optimality.

Therefore the optimal solution is $x^2 = (8, 0, 22, 8, 0)^t$ and the corresponding optimal value of the objective function is $f(x^2) = -1.227$.

4.2 Methods of Charnes and Cooper

Charnes and Cooper (1962) solved linear fractional programming by using one additional constraint and one additional non-linear variable.

Definition 4.8 *The problem*

$$\begin{array}{ll}
\textit{Minimum} & f(x) = \dfrac{c^t x + \alpha}{d^t x + \beta} \\
\textit{Subject to} & S = \{x : Ax = b, x \geq 0\}
\end{array}$$

is called regular if S is non empty, $f(x)$ is not constant and if there exist $M > 0$ such that $0 < d^t x + \beta < M$ for all $x \in S$, where $S \subset E$ and $E = \{x \in R^n : x \geq 0, d^t x + \beta > 0\}$.

Now, consider the problem of finding $x = (x_1, x_2, \cdots, x_n)^t$

$$\begin{aligned} \text{Minimum} \quad & f(x) = \frac{c^t x + \alpha}{d^t x + \beta} \qquad (P^1) \\ \text{Subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Where x, c and d are column of vectors all having n components, A is matrix of rank m, b is a columns of vectors having m components, α and β are real scalar constants.To avoid technical difficulty we assume that the constraints set S is regular i.e the set S of the feasible solution set is non-empty and bounded, and the denominator $d^t x + \beta$ is strictly positive throughout the constraint set.
So we can change linear fractional programming in to a linear programming by the following manner,

$$\begin{aligned} \text{Minimum} \quad & f(x) = \frac{c^t x + \alpha}{d^t x + \beta} \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \text{Minimum} \quad & (c^t \frac{x}{d^t x + \beta} + \alpha \frac{1}{d^t x + \beta}) \\ \text{Subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Let $w = \frac{1}{d^t x + \beta}$ and $z = xw$. Then the original problem becomes

$$\begin{aligned} \text{Minimum} \quad & (c^t z + \alpha w) \\ \text{Subject to} \quad & Az - bw = 0 \\ & z, w \geq 0 \end{aligned}$$

Now we have to minimize $c^t z + \alpha w$ with the change of variables $z = xw$, this becomes embedded in the following linear program of finding $z = (z_1, z_2, \cdots, z_n)$ and w in the following problem

$$\begin{aligned} \text{Minimum} \quad & c^t z + \alpha w = g(z, w) \qquad (P^2) \\ \text{Subject to} \quad & d^t x + \beta = 1 \\ & Az - bw = 0 \\ & z, w \geq 0 \end{aligned}$$

Lemma 4.9 *Every z, w satisfying the constraints*

$$\begin{aligned} d^t x + \beta &= 1 \\ Az - bw &= 0 \\ z, w &\geq 0 \end{aligned}$$

, has $w > 0$

Proof: Suppose $w = 0$, that is $(z, 0)$ is feasible and z must be different from 0, this means $x + \partial z$ would be feasible for the original problem for all $\partial > 0$ and any feasible x

$$\begin{aligned} A(x + \partial z) &= b \\ Ax + \partial Az &= b \\ Ax &= b \end{aligned}$$

This contradicts S bounded. Therefore $w > 0$. This completes the proof.
The following property extends to the case when (P^1) is regular a similar result obtained by charnes and cooper under the supposition that S is a bounded non empty set.

Theorem 4.10 *If the problem P^1 is regular, then for any feasible solution (z, w) of the problem P^2 there exist a feasible solution $x \in S$ of the problem P^1 such that $x = \frac{z}{w}$ and $f(x) = g(z, w)$, and the converse is also true.*

Proof: Let $w = \frac{1}{d^t x + \beta}$ and $z = xw$ or $x = \frac{z}{w}$

$$\begin{aligned} f(x) = \frac{c^t x + \alpha}{d^t x + \beta} \quad &\Leftrightarrow \quad c^t \frac{x}{d^t x + \beta} + \alpha \frac{1}{d^t x + \beta} \\ &\Leftrightarrow \quad c^t xw + \alpha w \\ &\Leftrightarrow \quad c^t z + \alpha w \\ &\Leftrightarrow \quad g(z, w) \end{aligned}$$

x is any feasible solution to P^1

$$\begin{aligned} Ax = b, x \geq 0 \quad &\Leftrightarrow \quad A(\frac{z}{w}) = b \text{ since } z = xw \\ &\Leftrightarrow \quad Az = bw \\ &\Leftrightarrow \quad Az - bw = 0 \end{aligned}$$

(z, w) is also a feasible solution to P^2. This completes the proof.
Next we need an auxiliary result which establish the relationship between problem P^1 and P^2 that generalizes for regular linear fractional programming a result obtained by charnes-cooper in the case when the feasible set is bounded and non empty.

Theorem 4.11 *If the problem is regular, then the following statement holds*
The problem P^1 and P^2 have both optimal solution and its optimal value are equal and finite. Moreover, if (z^*, w^*) is an optimal solution of P^2, then $x^* = \frac{z^*}{w^*}$ is an optimal solution for P^1 and conversely if $x' = \frac{z'}{w'}$ is an optimal solution of P^1, then there exist an optimal solution (z', w') of P^2 such that $z' = x'w'$.

Proof:($\Rightarrow$:) Given (z^*, w^*) is an optimal solution to P^2 by theorem 6, there exist is a feasible solution x^* such that $Ax^* = b$ and $x^* \geq 0$. We have to show $x^* = \frac{z^*}{w^*}$ is an optimal solution. Let x is a feasible solution such that $Ax^* = b$ and $x^* \geq 0$. Since $d^t x + \beta > 0$ by assumption and the vector (z, w) is a feasible solution to P^2. Where

$$z = \frac{x}{d^t x + \beta} \text{ and } t = \frac{1}{d^t x + \beta} \qquad (1)$$

Since (z^*, w^*) is an optimal solution to P^2.

$$c^t z * + \alpha w * \leq c^t z + \alpha w \qquad (2)$$

By substitution for z^*, z, w in (2) the equation become

$$w^*(c^t x^* + \alpha) \leq \frac{c^t x + \alpha}{d^t x + \beta}$$

The result immediately follows by dividing the left hand side by $1 = d^t z^* + \beta w^*$

$$\frac{c^t x^* + \alpha}{d^t x^* + \beta} \leq \frac{c^t x + \alpha}{d^t x + \beta}$$

Therefore x^* is an optimal solution for P^1.
($\Leftarrow$:) Given $x' = \frac{z'}{w'}$ is an optimal solution to P^1 and also a feasible solution. By theorem 6, there exists a feasible solution (z', w') such that $Az' - bw' = 0$ and $z' \geq 0$, $w > 0$. We have to show (z', w') is an optimal solution.
Let (z, w) is a feasible solution such that $Az - bw = 0$ and $z \geq 0$, $w > 0$, Where $z = \frac{x}{d^t x + \beta}$ and $w = \frac{1}{d^t x + \beta}$
Since x' is an optimal solution to a linear fractional program

$$\begin{aligned}
\frac{c^t x' + \alpha}{d^t x' + \beta} &\leq \frac{c^t x + \alpha}{d^t x + \beta} \\
c^t \frac{x'}{d^t x' + \beta} + \alpha \frac{1}{d^t x' + \beta} &\leq c^t \frac{x}{d^t x + \beta} + \alpha \frac{1}{d^t x + \beta} \\
c^t \frac{1}{d^t x' + \beta}(\frac{z'}{w'}) + \alpha \frac{1}{d^t x' + \beta}(\frac{w'}{w'}) &\leq c^t \frac{x}{d^t x + \beta} + \alpha \frac{1}{d^t x + \beta} \\
\frac{c^t z'}{(d^t x' + \beta) w'} + \frac{\alpha w'}{(d^t x' + \beta) w'} &\leq c^t \frac{x}{d^t x + \beta} + \alpha \frac{1}{d^t x + \beta} \\
c^t z' + \alpha w' &\leq c^t z + \alpha w
\end{aligned}$$

Since $(d^t x' + \beta) w' = d^t x' w' + \beta w' = 1$. Therefore, (z^*, w^*) is an optimal solution to a linear programming.This completes the proof.
To summarize, we have shown that a fractional linear program could be solved by a linear programming problem with one additional non-linear variable and one additional constraint.
Now we have to see numerical example of linear fractional programming problem solved by using method of Charnes and Cooper.

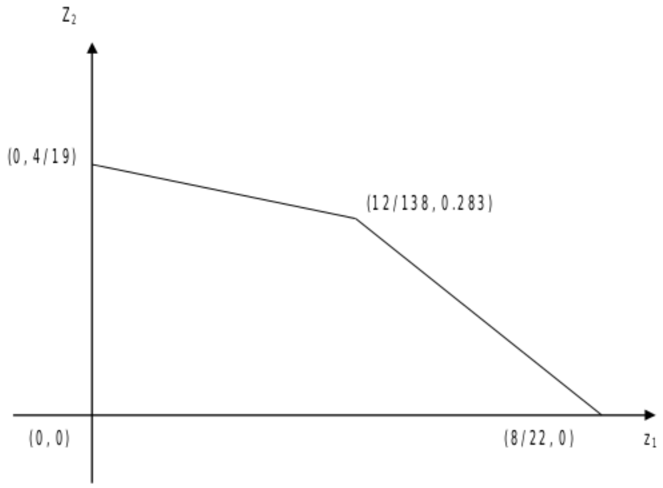
Example 4.12 *Consider the following problem*

$$\begin{array}{lll}
\textit{Minimum} & \dfrac{-4x_1 + 3x_2 + 5}{2x_1 + 4x_2 + 6} & (P^1) \\
\textit{subject to} & -2x_1 + x_2 \leq 6 & \\
 & x_2 \leq 8 & \\
 & 2x_1 + x_2 \leq 16 & \\
 & x_1, x_2 \geq 0 &
\end{array}$$

Solution: Since $(0, 0)$ is a feasible point in the above problem the denominator of the objective function is strictly positive over the entire feasible region. Now we have to minimize the following problems.
Let $z_1 = x_1 w$ and $z_2 = x_2 w$ (1)

$$\begin{array}{lll}
\text{Minimum} & -4z_1 + 3z_2 + 5w & (P^2) \\
\text{Subject to} & -2z_1 + z_2 - 6w \leq 0 & \\
 & z_2 - 8w \leq 0 & \\
 & 2z_1 + z_2 - 16w \leq 0 & \\
 & 2z_1 + 4z_2 + 16w = 1 & \\
 & z_1, z_2 \geq 0, w > 0 &
\end{array}$$

Graphs of constraints of the above problem is as The extreme points are $(0, 0)$, $(0, \frac{4}{19})$, $(\frac{12}{138}, 0.283)$ and



$(\frac{8}{22}, 0)$. The objective function value at those points is 0.833, 0.763, 0.2461 and -1.2777, respectively. Hence the optimal solution is $(\frac{8}{22}, 0)$ and the corresponding optimal value is -1.227.
Now first we have to add slack variables of y_1,y_2 and y_3, artificial variable y_4 and M is big number to change in to a standard linear optimization problem P^2 problem became

$$\begin{array}{ll}
\text{Minimum} & -4z_1 + 3z_2 + 5w + My_4 \\
\text{subject to} & -2z_1 + z_2 - 6w + y_1 \qquad\qquad\qquad \leq 0 \\
 & 1cmz_2 - 8w + \qquad y_2 \qquad\qquad \leq 0 \\
 & 2z_1 + z_2 - 16w + \qquad\qquad y_3 \qquad\qquad \leq 0 \\
 & 2z_1 + 4z_2 + 16w + \qquad\qquad\qquad y_4 \qquad = 1 \\
 & z_1, z_2, y_1, y_2, y_3, y_4 \geq 0, \text{ and } w > 0
\end{array}$$

Then the tableau is given by

	d_N	-4	3	5		
d_B	$y_B \nwarrow y_N$	z_1	z_2	w	b	$\triangle$
0	y_1	-2	1	-6	0	-
0	y_2	0	1	-8	0	-
-4	y_3	2	1	-16	0	-
5	y_4	2	4	6	1	$\frac{1}{6}$
	$\triangle\ z$	-4-2M	3-4M	5-6M		

Since min $\triangle\ z = 5 - 6M < 0$, increase w and decreasing y_4 to zero.
Then the new tableau is given by

	d_N	-4	3	M		
d_B	$y_B \nwarrow y_N$	z_1	z_2	y_4	b	$\triangle$
0	y_1	0	5	1	1	-
0	y_2	$\frac{8}{3}$	$\frac{19}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{1}{2}$
0	y_3	$\frac{22}{3}$	$\frac{35}{3}$	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{8}{22}$
5	w	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
	$\triangle\ z$	$\frac{-17}{5}$	$\frac{-1}{3}$	0	0	

Since min $\triangle z = \frac{-17}{5} < 0$, increase z_1 and decreasing y_3 to zero.
Then the new tableau is given by

	d_N	0	3	0		
d_B	$y_B \nwarrow y_N$	y_3	z_2	y_4	b	$\triangle$
0	y_1	0	5	1	1	-
0	y_2	$\frac{-8}{22}$	$\frac{46}{22}$	$\frac{8}{22}$	$\frac{8}{22}$	$\frac{1}{2}$
-4	y_3	$\frac{3}{22}$	$\frac{35}{22}$	$\frac{8}{22}$	$\frac{8}{22}$	$\frac{8}{22}$
5	w	$\frac{-1}{22}$	$\frac{3}{22}$	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{2}$
	$\triangle z$	$\frac{17}{22}$	$\frac{191}{22}$	$\frac{27}{22} + M$	-1.27227	

All $\triangle z \geq 0$. This is the condition of optimality.

Therefore the optimal feasible solution is $(\frac{8}{22}, 0, \frac{1}{11}, \frac{8}{22}, 0, 0)$. Therefore $z_1 = \frac{8}{22}$, $z_2 = 0$, and $w = \frac{1}{22}$ from (1). So $x_1 = \frac{z_1}{w} = 8$ and $x_2 = \frac{z_2}{w} = 0$, i.e $(8, 0)$ is the extreme optimal solution to a linear fractional problem and the corresponding optimal value is -1.227.

5 Conclusion

This paper focused on Fractional Programming in particular Linear Fractional Programming. Such kind of programming problems has recently been a subject of wide interest in non linear programming. Linear Fractional Programming problem have a global solution because of convexity property and can be solved by different methods. Convex simplex method, Gilmore and Gomory procedure, and Charnes and Cooper method are some of the methods that solve Linear Fractional Programming.
Each of the method mention above is its own property. Convex simplex method is almost identical to reduced gradient method, the difference is only it changes one non basic variables and modifying basic variables, while reduced gradient method is by generating feasible direction and convex simplex method is reduced to simplex method whenever the objective function is linear or a linear portion of the objective function is encounter, while reduced gradient method may not so reduced. Convex simplex method is identical to method of Gilmore and Gomory in a selection of basic feasible solution and pivot points when convex simplex method is applying to Linear Fractional Programming. On the other hand, the advantage of Charnes and Cooper algorithm is reducing Linear Fractional Programming in to Linear Programming to facilitating finding of solution by using simplex method but the disadvantage is it needs one more additional variables and one constraints. Lastly convex simplex method is behaves generality like the linear simplex method whenever any linear portion of the objective function are encounter.
The method explained here will be useful in the solution of economic problems in which the different economic activities utilize fixed resources in proportion to the level of there values.

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